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Abstract

In this paper we reconsider the concept of Berge equilibrium. In a recent work, [Colman, A. M., Körner, T., Musy, O. and Tazdaït, T. [2011] Mutual support in games: Some properties of Berge equilibria, *Journal of Mathematical Psychology* **55**, 166–175]. proposed a correspondence for two-player games between Berge and Nash equilibria by permutation of the utility functions. We define here more general transformations of games that lead to a correspondence with Berge and Nash equilibria and characterize all such transformations.

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1 Introduction

A burgeoning literature has focused on the Berge-Vaisman equilibrium (Abalo and Kostreva (2004, 2005); Colman et al. (2011); Courtois et al. (2011); Larbani and Nessah (2008); Musy et al. (2012); Nessah et al. (2007)). This concept captures the mutual support behavior in the normal form games. Furthermore, it is not a refinement of the Nash equilibrium. It thus fills a gap in the non-cooperative game theory.

Among different results that have been highlighted, there is one that will particularly come under scrutiny in this paper. Colman et al. (2011) established a correspondence between Berge-Vaisman and Nash equilibria. These authors show that the set of Berge-Vaisman equilibria of a two-player game is the set of Nash equilibria obtained by permutation of the utility functions of both players. In general, such a transformation may be feasible for one game, but it is then specific of the game. The aim of this paper is to examine whether there exists a general transformation of Berge-Vaisman equilibrium in terms of Nash equilibrium that could be valid not for one specific game, but for all games. Accordingly, we give all the possible transformations for two-player games. We also show that for n-player games, with $n > 2$, there is no transformation that links Berge-Vaisman and Nash equilibria.

That a transformation between Nash and Berge-Vaisman equilibria may exist is not a fantasy from the authors. The concept of Berge-Vaisman equilibrium is sometimes dismissed, on the basis that it is simply Nash equilibrium "up to a transformation". The confusion may have arisen because in the most studied two-player case such a transformation exists. We clear away this confusion. When one is acquainted with Berge-Vaisman equilibrium, it is not surprising that a "transformation" does not exist. Together with precise definitions, we provide here a comprehensive proof of this fact.

2 Berge-Vaisman Equilibrium and General Transformation

Consider the following non cooperative game in normal form:

$$G = \langle I, (S_i)_{i\in I}, (u_i)_{i\in I}\rangle, \tag{1}$$

where $I = \{1, \dots, n\}$ denotes the set of players (with $n \geq 2$), S_i the non-empty strategy set of player i, and u_i her utility function. This utility function $u_i : S \to \mathbb{R}$ is defined on $S = \prod_{i\in I} S_i$, where S is the set of all strategy profiles and s_{-i} is the strategy profile $(s_1, \dots s_{i-1}, s_{i+1} \dots s_n) \in S_{-i} = \prod_{j\neq i} S_j$. We start with the definition of Nash and Berge-Vaisman equilibria.

Definition 1 (Nash (1950)) *A feasible strategy profile $\overline{s} \in S$ is said to be a Nash equilibrium of the game G if, for any player $i \in I$, and any $s_i \in S_i$, we have:*

$$u_i(s_i, \overline{s}_{-i}) \leq u_i(\overline{s}).$$

A Nash equilibrium is defined as a strategy profile in which no agent, taking the strategies of the other players as given, wishes to change her strategy choice.

Definition 2 (Zhukovskii (1985)) *A feasible strategy profile $\overline{s} \in S$ is a Berge-Vaisman equilibrium of the game G if, for any player $i \in I$, and any $s_{-i} \in S_{-i}$, we have:*

$$u_i(\overline{s}_i, s_{-i}) \leq u_i(\overline{s}).$$

This definition means that, when a player $i \in I$ plays her strategy $\overline{s}_i$ from the Berge-Vaisman equilibrium $\overline{s}$, she obtains her higher utility when the remaining players willingly play the strategy $\overline{s}_{-i}$ from the Berge-Vaisman equilibrium.

These two equilibria are very different from one another. Yet, Colman et al. (2011) found a correspondence between the two equilibrium concepts. By linking the game $G = \langle \{1,2\}, (S_1, S_2), (u_1, u_2) \rangle$ to the game $\tilde{G} = \langle \{1,2\}, (S_1, S_2), (v_1, v_2) \rangle$ with $v_1 = u_2$ and $v_2 = u_1$, these authors proved the following theorem[1].

Theorem 1 (Colman et al. (2011)) *A strategy profile $\overline{s} \in S$ is a Berge-Vaisman equilibrium of the game G if and only if it is a Nash equilibrium of the game $\tilde{G}$.*

We recall here the proof of this theorem for sake of self-consistency.

Proof. This is a simple play with the definition, noting that $S_{-1} = S_2$ and $S_{-2} = S_1$.

Suppose s^* is a Berge equilibrium of the game G. Then, by definition,

$$u_i(s_i^*, s_{-i}) \leq u_i(s^*), \forall\, i \in N, \ \forall\, s_{-i} \in S_{-i},$$

That is to say $u_1(s_1^*, s_2) \leq u_1(s^*), \ \forall\, s_2 \in S_2$ and $u_2(s_1, s_2^*) \leq u_2(s^*), \ \forall\, s_1 \in S_1$.

Since $u_1 = v_2$ and $u_2 = v_1$, $v_1(s_1, s_2^*) \leq v_1(s^*), \ \forall\, s_1 \in S_1$ and $v_2(s_1^*, s_2) \leq v_2(s^*), \forall\, s_2 \in S_2$. Therefore

$$v_i(s_i, s_{-i}^*) \leq v_i(s^*), \ \forall\, i \in N, \ \forall\, s_i \in S_i$$

It follows that s^* is a Nash equilibrium of the game $\tilde{G}$. The converse that if s^* is a Nash equilibrium in $\tilde{G}$, then it is a Berge equilibrium in G is proved in the same way, and the required result follows. □

To propose a more general result, we introduce the following definition.

Definition 3 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a function, F is a transformation of the Berge-Vaisman equilibrium concept to the Nash equilibrium concept, if for all n-player games, the set of Berge-Vaisman equilibrium of the game*

$$G \ = \ \langle I, (S_i)_{i \in I}, (u_i)_{i \in I} \rangle,$$

is the same as the set of Nash equilibrium in the transformed game

$$F(G) \ = \ \langle I, (S_i)_{i \in I}, (F(u)_i)_{i \in I} \rangle,$$

where the utility of player i is $F_i(u_1, \ldots, u_n)$.

[1] Note that Larbani and Nessah (2008) identified another type of link. In particular, they showed that a Berge-Vaisman equilibrium can also be a Nash equilibrium under certain conditions.

We call such an F simply an n-transformation. The game $F(G)$ is called the transformed game of G. We can formulate the theorem of Colman et al. (2011) with this definition.

Theorem 2 (Theorem 1 restated) *The function $F(x,y) = (y,x)$ is a 2-transformation.*

The paper now characterizes all the n-transformations. We begin with a general lemma.

Lemma 1 *Suppose that F is an n-transformation ($n \geq 2$), then F_i does not depend on u_i.*

Proof. We want to prove that for all $u_{-i} \in \mathbb{R}^{n-1}$, F_i is constant in u_i. By symmetry, we put $i = 1$. Suppose our lemma is false, then there exist u_{-1} and $u_1, u_1' \in \mathbb{R}$, such that $F_1(u_1, u_{-1}) \neq F_1(u_1', u_{-1})$. Then we define the n-player game by the strategy set $S_1 = \{s_1, s_1'\}$ and $S_{-1} = \{s_{-1}\}$, and the utilities $u_1(s_1, s_{-1}) = u_1$, $u_1(s_1', s_{-1}) = u_1'$ and $u_{-1}(s_1, s_{-1}) = u_{-1}, u_{-1}(s_1', s_{-1}) = u_{-1}$. The two strategy profiles of this game are Berge-Vaisman equilibria, however the transformed game has just one Nash equilibrium (it is (s_1, s_{-1}) if $F_1(u_1, u_{-1}) > F_1(u_1', u_{-1})$, (s_1', s_{-1}) if $F_1(u_1, u_{-1}) < F_1(u_1', u_{-1})$). This is a contradiction with the assumption that F is a transformation. $\square$

2.1 The case $n = 2$

Now we restrict ourselves to the case $n = 2$. According to the lemma, the transformation F takes the form $F(u_1, u_2) = (F_1(u_2), F_2(u_1))$.

Lemma 2 *Suppose that F is a 2-transformation, then F_1 is a strictly increasing function of u_2.*

Proof. Suppose it is not the case. Then, there exist $u_2, u_2' \in \mathbb{R}$ such that $u_2 < u_2'$ but $F_1(u_2) \geq F_1(u_2')$. Consider the 2-player game defined by the strategy set $S_1 = \{s_1, s_1'\}$ and $S_2 = \{s_2\}$, and the utility $u_1(s_1, s_2) = u_1$, $u_1(s_1', s_2) = u_1$ and $u_2(s_1, s_2) = u_2, u_2(s_1', s_2) = u_2'$, where u_1 is any real number. The strategy profile (s_1', s_2) is the unique Berge-Vaisman equilibrium, whereas in the transformed game (s_1, s_2) is a Nash equilibrium (it is unique if $F_1(u_2) > F_1(u_2')$, if $F_1(u_2) = F_1(u_2')$, then (s_1', s_2) is also a Nash equilibrium). This contradicts the assumption that F is a transformation, i.e. that Nash equilibria of the transformed game are exactly the Berge-Vaisman equilibria of the original game. $\square$

Theorem 3 *All the 2-transformations have the form $F(u_1, u_2) = (F_1(u_2), F_2(u_1))$, where F_1 and F_2 are strictly increasing functions.*

Proof. From the lemma 2, by symmetry, we have also proven that F_2 is a strictly increasing function of u_1. So the conditions of the theorem are necessary for F to be a transformation. To prove that they are also sufficient, we proceed in two steps. By theorem 1, permuting the utility function of the two players is a 2-transformation. Then changing the utility of each player by increasing functions does not change the set of equilibria. So the functions with the form specified in the theorem are indeed transformations. $\square$

For two-player games, the concept of Berge-Vaisman equilibrium is reducible to the concept of Nash equilibrium, given that the utilities of the players be redefined. The above theorem precisely specifies the set of transformations for two-player games. Let us now move to the case where the number of players is greater than 2.

2.2 The case $n > 2$

Let F be an n-transformation. We have already proven that F_i does not depend on u_i. We proceed to the *reductio ad absurdum* process by proving the following lemma.

Lemma 3 *Suppose that F is an n-transformation. Then F_1 does not depend on u_2.*

Proof. Assuming the lemma is not valid, there exists $u_{-2}, u_2, u'_2 \in \mathbb{R}$, such that $F_1(u_2, u_{-2}) \neq F_1(u'_2, u_{-2})$. We split u_{-2} in $(u_1, u_{-1,-2})$ so that $u_{-1,-2} \in \mathbb{R}^{n-2}$ (if $n = 3$, this is just u_3). Consider the n-player game defined by the strategy set $S_1 = \{s_1\}$, $S_2 = \{s_2\}$, and $S_{-1,-2} = \{s_{-1,-2}, s'_{-1,-2}\}$. The utilities for players other than 2 are constant $u_1 = u_1$, $u_{-1,-2} = u_{-1,-2}$, but for player 2 $u_2(s_1, s_2, s_{-1,-2}) = u_2, u_2(s_1, s_2, s'_{-1,-2}) = u'_2$. Note that the definition of u_2 is valid precisely because $n > 2$. In the transformed game, the two strategy profiles $(s_1, s_2, s_{-1,-2})$ and $(s_1, s_2, s'_{-1,-2})$ are Nash equilibria, whereas in the initial game, only one strategy profile is a Berge-Vaisman equilibrium (it is $(s_1, s_2, s_{-1,-2})$ if $u_2 > u'_2$). This contradicts our general assumption that F is a transformation and proves the lemma. □

We are now ready to show that an n-transformation cannot exist if $n > 2$.

Theorem 4 *No n-transformation exists when $n > 2$.*

Proof. In the proof of the previous lemma, the place of 2 was irrelevant, the only important thing was that there exists at least a third player (distinct from 1 and 2). So actually F_1 does not depend on any u_i for all i (recall our previous lemma). But it is irrelevant whether it is 1 or another player (the proof is the same if one replaces 1 by i and 2 by j). So we have the result that F is constant. But a constant cannot be a transformation, because there are games where not all strategy profiles are Berge-Vaisman equilibria (*e.g.* the game constructed in the proof of lemma 3). □

3 Illustration Example

A key ingredient of the proof is the requirement of the transformation to be universable. If this condition is relaxed, a transformation may exist, as the example will show.

Example 1 Consider the following n-players (assume that $n > 2$) on the unit square $S_i = [0, 1]$, for each $i \in I = \{1, ..., n\}$. For player $i \in I$ and $s = (s_1, ..., s_n) \in S = [0, 1]^n$, the payoff functions are:

$$u_i(s) = \sum_{j=1}^{n} a_{i,j} s_j,$$

where $a_{i,j}$ is constant in $\mathbb{R}$, for each $i, j \in I$. This game is bounded, compact, convex, continuous and quasiconcave.

For each player $i \in I$, denote by I_{-j} all players rather than player j ($I_{-j} = \{1, \ldots, j-1, j+1, \ldots, n\}$). The following proposition characterizes the existence of Berge-Vaisman equilibrium.

Proposition 1 *The considered game possesses a Berge-Vaisman equilibrium if and only if for each $j \in I$, $sign(a_{i,j}) = sign(a_{h,j})$,* [2] *for each $i, h \in I_{-j}$.*

[2] $sign(x) = \begin{cases} + & \text{if } x \geq 0 \\ - & \text{if } x \leq 0. \end{cases}$

Proof. Let $\overline{s} \in S$ be a Berge-Vaisman equilibrium of the game. Fix any $j \in I$, and let $i \in I_{-j}$. Since $\overline{s}$ is a Berge-Vaisman equilibrium, then

$$u_i(\overline{s}_i, s_{-i}) \leq u_i(\overline{s}), \text{ for each } s_{-i} \in S_{-i}$$

Let $s_{-i} = (s_j, \overline{s}_{-\{i,j\}})$ for any $s_j \in S_j$. This is possible, since $j \neq i$. Thus,

$$a_{i,j}s_j \leq a_{i,j}\overline{s}_j, \text{ for each } s_j \in S_j \text{ and } i \neq j$$

If $a_{i,j} > 0$, then necessarily $\overline{s}_j = 1$. If $a_{i,j} < 0$, then necessarily $\overline{s}_j = 0$. If $a_{i,j} = 0$, there is no condition on $\overline{s}_j$. Obviously, $\overline{s}_j$ does not depend on $i \in I_{-j}$,, therefore $sign(a_{i,j}) = sign(a_{h,j})$, for each $i, h \in I_{-j}$.

Conversely, suppose the conditions holds, then for $j \in I$ define $\overline{s}_j$ to the (common) sign of the $a_{i,j}, i \in I_{-j}$ (i.e. 0 is at least one of the $a_{i,j}$ is negative, 1 if at least one is positive, any number in S_j if all are null). Then the strategy profile $\overline{s}$ is a Berge-Vaisman equilibrium of the game. □

This fully characterizes the Berge-Vaisman equilibria of the game with linear utilities. We now do the same for the Nash equilibria.

Proposition 2 *1) The game $\langle I, S, (u_i)_{i \in I} \rangle$ possesses an unique Nash equilibrium if and only if for each $i \in I$, $a_{i,i} \neq 0$.*

2) The game $\langle I, S, (u_i)_{i \in I} \rangle$ possesses an infinity number of Nash equilibria if and only if there exists $i_0 \in I$, so that $a_{i_0,i_0} = 0$.

Proof. Since the game $\langle I, S, (u_i)_{i \in I} \rangle$ is compact, convex, continuous and quasiconcave, then it has a Nash equilibrium. Let $\overline{s} \in S$ be a Nash equilibrium of the game. Then for each i, and for each $s_i \in S_i$, we have $u_i(s_i, \overline{s}_{-i}) \leq u_i(\overline{s})$. By linearity of u_i, we obtain then

$$a_{i,i}s_i \leq a_{i,i}\overline{s}_i, \text{ for each } i \in I.$$

If $a_{i,i} > 0$, this is possible only if $\overline{s}_i = 1$; if $a_{i,i} < 0$, this is possible only if $\overline{s}_i = 0$. If $a_{i,i} = 0$, there is no condition on $\overline{s}_i$.

Conversely, define $\overline{s}_i$ according to the sign of $a_{i,i}$ (i.e. $\overline{s}_i = 1$ when $a_{i,i} > 0$; $\overline{s}_i = 0$ when $a_{i,i} < 0$; any $s_i \in S_i$ when $a_{i,i} = 0$). Then the strategy profile $\overline{s}$ is a Nash equilibrium. □

We can now give two examples of a transformation between Berge-Vaisman and Nash equilibrium.

Proposition 3 *Consider the class of all games with linear utilities, such that for each $j \in I$, for $i, h \in I_{-j}$ $sign(a_{i,j}) = sign(a_{h,j})$. Restricts further the class to the game such that $a_{i,i}$ has the same sign of $a_{h,i}$, $h \neq i$, in a strict sense, that is $a_{i,i}$ is positive (resp. negative) if at least one $a_{h,i}$, $h \neq i$ is positive (resp. negative), $a_{i,i}$ is null if all $a_{h,i}$, $h \neq i$ are null. The identity is a transformation on this class.*

Proof. The first condition ensures by Proposition 1 that a Berge-Vaisman equilibrium exists. The second condition ensures that Nash and Berge-Vaisman equilibrium are the same by the characterization of equilibrium strategy profiles of Propositions 1-2. So the identity is a transformation of game. □

Let call F the function that maps a vector $x \in \mathbb{R}^n$ to the diagonal vector of the sum of its components.

Proposition 4 *Consider the class of all games with linear utilities, such that for each $j \in I$, for $i, h \in I_{-j}$ $sign(a_{i,j}) = sign(a_{h,j})$ and $(\sum_i a_{i,j}) \left(\sum_{i \neq j} a_{i,j}\right) > 0$. Then F is a transformation on this class of games.*

Proof. Take a game G in the class. By the conditions that all signs of coefficients (off the diagonal) of a column are the same, there is a Berge-Vaisman equilibrium. This equilibrium is unique for not all the coefficients are null by the extra-condition. By Proposition 1, the Berge-Vaisman strategy profile is uniquely determined by the sign of the columns. The transformed game $F(G)$ still has linear utilities, with coefficients $b_{j,j} = \sum_i a_{i,j}$. The extra-condition $(\sum_i a_{i,j}) \left(\sum_{i \neq j} a_{i,j}\right) > 0$ ensures that $b_{j,j}$ has the same sign of the column j of the original game. So the Nash equilibrium of the transformed game is the same as the Berge-Vaisman equilibrium of the original game. □

4 Conclusion

In this paper we have characterized all the transformation of games between Nash and Berge-Vaisman equilibria. For 2-player games, Berge-Vaisman equilibrium is linked to Nash equilibrium by a permutation of utilities pertaining to changing utilities by strictly increasing functions. For n-player games, with $n > 2$, the concept of Berge-Vaisman equilibrium is not transformable to the concept of Nash equilibrium by redefining the utilities of the players. This result stimulates further research on Berge-Vaisman equilibrium when there are more than two players.

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